PHYSICAL REVIEW VOLUME 132, NUMBER 5 1 DECEMBER 1963

Forward Pion-Pion Scattering in the $\lambda\phi^4$ Theory*

M. BAKER† AND I. J. MUZINICH
University of Washington, Seattle, Washington
(Received 17 July 1963)

An exact expression is obtained for the sum of the bubble diagrams for forward pion-pion scattering in the $\lambda\phi^4$ theory, with zero mass internal pions. The exact solution of the problem for massless internal pions follows naturally from the solution of the Bethe–Salpeter equation, with the bubble-exchange kernel, which is the asymptotic limit of the equation for pions with finite mass.

INTRODUCTION

IN this work we obtain an exact solution to the forward pion-pion scattering Bethe-Saltpeter equation arising from the sum of bubble-exchange diagrams in the $\frac{1}{4}\lambda\phi^4$ theory with massless pions. In the high-energy limit, $E\rightarrow\infty$, the forward scattering amplitude $f(E)$ takes the form

$$f(E)\sim\frac{E^{(1+2\bar{\lambda})}-1}{(\ln E)^{1/2}}, \qquad E\rightarrow\infty \tag{I.1}$$

where $\bar{\lambda}^2=\lambda^2/32\pi^4$.

This model is of interest for the following reasons:

(1) The integral equation which we solve is of a non-Fredholm type. The model thus represents at least one aspect of a general field theory which is not present in potential theory.

(2) Most of the current results on the high-energy limit of scattering amplitudes have been obtained by summing the most singular terms in each order of perturbation theory.[1] In particular this technique has been applied by Sawyer[1] to the above model. This summation procedure yields explicit expressions for high-energy scattering amplitudes only in the weak-coupling limit. We can then test the validity of such a procedure by comparing Sawyer's result with the weak-coupling limit of Eq. 1.

(3) The exact solution to this simple model may be useful to reduce a more general problem with the same singular behavior to a Fredholm integral equation.

The method we employ is due to Bjorken[2] who has recently investigated the properties of the full $\frac{1}{4}\lambda\phi^4$ theory with all propagators modified so as to produce the convergence of the unrenormalized perturbation theory. Bjorken separates the four-dimensional scattering Bethe-Saltpeter equation using Tschebyscheff polynomials, $C_n(\cos\theta)$. He then uses analytic continuation in the n plane in analogy with Regge's use of analytic continuation in the angular momentum plane.[3]

The present work was undertaken in order to see how Bjorken's results are modified when this method is applied to some of the simple renormalized diagrams of the unmodified ϕ^4 theory.

In Sec. II we review Bjorken's method with reference to the bubble exchange diagrams and note the non-Fredholm nature of the equation. In Sec. III, we see that by solving an equation whose kernel is the asymptotic limit of the original kernel we can reduce the original equation to a Fredholm equation. This asymptotic problem then leads us naturally to the zero mass equation. In Sec. IV we use the Watson-Sommerfield transformation to obtain an explicit integral representation for the solution to the mass zero forward scattering amplitude. In Sec. V this integral is evaluated in the high-energy limit.

II. THE BETHE-SALPETER EQUATION IN THE n PLANE

The Bethe-Salpeter equation for the forward scattering amplitude $A(\mathbf{kq})$ for the mesons of four momenta $\mathbf{k}$ and $\mathbf{q}$ due to bubble exchange diagrams (Fig. 1) is

$$A(\mathbf{kq})=B(\mathbf{kq})-i\int\frac{d^4k'}{(2\pi)^4}\frac{B(\mathbf{kk'})}{(k'^2+\mu^2)^2}A(\mathbf{k'q}), \tag{II.1}$$

where $B(\mathbf{kq})=B[-(\mathbf{k}-\mathbf{q})^2]+B[-(\mathbf{k}+\mathbf{q})^2]$ with

$$B(x)=(x-x_0)\frac{\lambda^2}{16\pi^2}\times\int_{4\mu^2}^{\infty}\frac{dx'(1-4\mu^2/x')^{1/2}}{(x'-x_0)(x'-x+i\epsilon)}+B(x_0), \tag{II.2}$$

x_0 is an arbitrary subtraction point. We use a space-like metric: $k^2=\boldsymbol{\kappa}^2-k_0^2$. Now $A(\mathbf{kq})$ is a function of k^2, q^2,

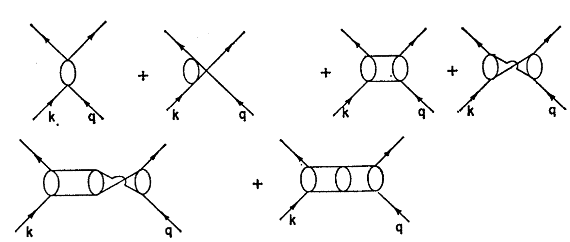

FIG. 1. Typical bubble-exchange diagrams.

* Supported, in part, by the U. S. Atomic Energy Commission under Contract A.T.(45-1)1388, Program B.

† Alfred P. Sloan Foundation Fellow.

[1] R. Sawyer, Phys. Rev. **131**, 1384 (1963); J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963); and P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) **22**, 263 (1963).

[2] J. D. Bjorken (to be published).

[3] T. Regge, Nuovo Cimento **18**, 947 (1960), and **14**, 951 (1959).

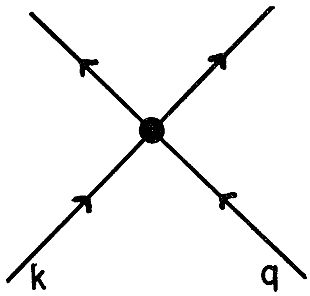

FIG. 2. Direct interaction diagram.

and $\mathbf{k}\cdot\mathbf{q}$. For physical values of $\mathbf{k}$ and $\mathbf{q}$, $k^2=q^2=-\mu^2$ and $-\mathbf{k}\cdot\mathbf{q}=\mu E$ where E is the laboratory energy and μ is the meson mass. We can define an angle θ_{kq} as

$$\cos\theta_{kq}=\frac{\mathbf{k}\cdot\mathbf{q}}{(k^2q^2)^{1/2}}, \quad \text{(II.3)}$$

$\cos\theta_{kq}>1$.

Following Bjorken, we analytically continue Eq. (II.1) to the region $k^2>0$, $q^2>0$, $\cos\theta_{kq}<1$. Then Eq. (II.1) becomes an equation over a four-dimensional Euclidean region with scalar products $k^2=\boldsymbol{\kappa}^2+k_0^2$.

$$A(k^2,q^2,\cos\theta_{kq})=B(k^2,q^2,\cos\theta_{kq}) + \int\frac{d^4k'}{(2\pi)^4}\frac{B(k^2,k'^2,\cos\theta_{kk'})}{(k'^2+\mu^2)^2}A(k'^2,q^2,\cos\theta_{k'q}). \quad \text{(II.4)}$$

The four-dimensional integral over k' in (II.4) can be written as

$$\int d^4k'=\int k'^3dk'd\Omega_{k'}$$

where $d\Omega_{k'}$ is the element of solid angle of a four-dimensional sphere.

We now expand $A(k^2,q^2,\cos\theta_{kq})$ in terms of the Tschebyscheff polynomials,

$$C_n(\cos\theta)=\sin(n+1)\theta/\sin\theta, \quad \text{(II.5)}$$

$$A(k^2,q^2,\cos\theta_{kq})=\sum_{n=0}^{\infty}A_n(k,q)C_n(\cos\theta_{kq}). \quad \text{(II.6)}$$

These polynomials have the property,

$$\frac{1}{2\pi^2}\int d\Omega_kC_n(\cos\theta_{ik})C_m(\cos\theta_{kf})=\frac{\delta_{nm}}{n+1}C_n(\cos\theta_{if}). \quad \text{(II.7)}$$

From Eqs. (II.4), (II.6), and (II.7) we get the following equations for $A_n(k,q)n\geq2$:

$$A_n(k,q)=B_n(k,q) + \frac{1}{8\pi^2(n+1)}\int_0^\infty\frac{k'^3dk'}{(k'^2+\mu^2)^2}B_n(k,k')A_n(k',q), \quad \text{for } n\geq2, \quad \text{(II.8)}$$

where

$$B_n(k,q)=\frac{2}{\pi}\int_0^\pi d\theta\sin^2\theta B(k^2,q^2,\cos\theta)C_n(\cos\theta). \quad \text{(II.9)}$$

From Eq. (II.2) we get the following expression for $B_n(k,q)$, for $n\geq2$:

$$B_n(k,q)=\frac{\lambda^2}{kq8\pi^2}\int_{4\mu^2}^\infty dx'\left(\frac{x'-4\mu^2}{x'}\right)^{1/2} \times\left\{\frac{1}{\left[\left(\frac{x'+k^2+q^2}{2kq}\right)^2-1\right]^{1/2}+\left(\frac{x'+k^2+q^2}{2kq}\right)}\right\}^{n+1} \quad n \text{ even}\geq2, \quad \text{(II.10)}$$

$B_n(k,q)=0$, n odd.

Equation (II.8) admits an iteration solution for $n\geq2$. Thus the divergence in Eq. (II.1) which is reflected in Eq. (II.8) by the divergence of B_0, contributes only to A_0. Likewise the direct pion-pion interaction diagram of Fig. 2, whose contribution to $B(\mathbf{kq})$ has not been considered, contributes only to A_0. Now the $n=0$ contribution to Eq. (II.6), is independent of E. Thus, aside from an infinite constant which is removed by renormalization we can write

$$A(k^2,q^2,\cos\theta_{kq})=\sum_{n=2}^{\infty}A_n(k,q)C_n(\cos\theta_{kq}), \quad \text{(II.11)}$$

where $A_n(k,q)$ satisfies Eq. (II.8) with $B_n(k,q)$ given by Eq. (II.10).

This procedure is identical to Bjorken's. The results will essentially be different, however, since the integral Eq. (II.8) is not of the Fredholm type. Thus, although Eq. (II.8) for the kernel $B_n(k,q)$ can be continued into the complex n plane as an analytic function of n for $\mathrm{Re}n>0$, the solution A_n of Eq. (II.8) will possess quite distinct analytic properties. If the usual Fredholm theory had been applicable, we could have inferred the analytic properties of A_n for $\mathrm{Re}n>0$ directly from the form of the Fredholm solution.[2,4]

It is the behavior of $B_n(k',k)$ in the region where k and k' are both large and comparable that produces the non-Fredholm nature of Eq. (II.10). Our approach then will be to find an equation with the same asymptotic properties in k'^2 which we can solve exactly.

III. THE ASYMPTOTIC EQUATION

Let $B_{0n}(k,q)$ be the limit of $B_n(k,q)$ as either k^2 or q^2 become large. The integral Eq. (II.10), in this limit is readily carried out and yields

$$B_{0n}(k,q)=\frac{\lambda^2}{8\pi^2}\left[\frac{1}{n}\left(\frac{k}{q}\right)^n-\frac{1}{n+2}\left(\frac{k}{q}\right)^{n+2}\right] \quad q>k,$$
$$=\frac{\lambda^2}{8\pi^2}\left[\frac{1}{n}\left(\frac{q}{k}\right)^n-\frac{1}{n+2}\left(\frac{q}{k}\right)^{n+2}\right] \quad k>q. \quad \text{(III.1)}$$

[4] L. Brown, D. Fivel, B. W. Lee, and R. Sawyer (to be published).

Equation (II.8) can then be replaced by the two equations whose sum is Eq. (II.8)

$$A_n(k,q)=I_n(k,q)+\frac{1}{(n+1)8\pi^2}\times\int_0^\infty \frac{dk'k'^3}{(k'^2+\mu^2)^2}B_{0n}(k',k)A_n(k',q)\,, \quad \text{(III.2)}$$

$$I_n(k,q)=B_n(k,q)+\frac{1}{(n+1)8\pi^2}\int_0^\infty \frac{dk'k'^3}{(k'^2+\mu^2)^2}\times[B_n(k',k)-B_{0n}(k',k)]A_n(k',q)\,. \quad \text{(III.3)}$$

We first solve Eq. (III.2). This yields an expression for $A_n(kq)$ of the form

$$A_n=I_n+\int K_nI_n\,, \quad \text{(III.4)}$$

where K_n is the resolvent kernel of Eq. (III.2.) Then from Eqs. (III.3) and (III.4), we get an equation for $A_n(kq)$ of the form

$$A_n=B_n+\int K_nB_n+\int (B_n-B_{0n})A_n+\int\int K_n(B_n-B_{0n})A_n\,. \quad \text{(III.5)}$$

We expect the Fredholm theory to be applicable to Eq. (III.5) since it's kernel involves $B_n(k,k')-B_{0n}(k,k')$. In particular the essential features of the solution due to the high k and k' behavior of the kernel (non-Fredholm nature) should be contained in the inhomogeneous term of Eq. (III.5) with B_n replaced by B_{0n}. That is, we expect

$$A_n=B_{0n}+\int K_nB_{0n} \quad \text{(III.6)}$$

plus corrections, which can be treated by the usual Fredholm method.

The integral Eq. (III.2) is easily reduced to an inhomogeneous fourth-order differential equation which we have not been able to solve. However, if we set $\mu=0$, the solution is elementary and is

$$A_n(k,q)=I_n(k,q)+\frac{\lambda^2}{2[(n+1)^2+\bar{\lambda}^2]^{1/2}}\times\left\{\frac{1}{\alpha_1}\int_0^k\frac{dk'}{k'}\left(\frac{k'}{k}\right)^{\alpha_1}I_n(k,'q)-\frac{1}{\alpha_2}\int_0^k\frac{dk'}{k}\left(\frac{k'}{k}\right)^{\alpha_2}I_n(k',q)+\frac{1}{\alpha_1}\int_k^\infty\frac{dk'}{k'}\left(\frac{k}{k'}\right)^{\alpha_1}I_n(k',q)-\frac{1}{\alpha_2}\int_k^\infty\frac{dk'}{k'}\left(\frac{k}{k'}\right)^{\alpha_2}I_n(k',q)\right\}, \quad \text{(III.7)}$$

where

$$\alpha_1=[(n+1)^2+1-2((n+1)^2+\bar{\lambda}^2)^{1/2}]^{1/2},$$
$$\alpha_2=[(n+1)^2+1+2((n+1)^2+\bar{\lambda}^2)^{1/2}]^{1/2}. \quad \text{(III.8)}$$

By direct substitution we can verify that Eq. (III.7) satisfies Eq. (III.2) with $\mu=0$ provided $I(k',k)$ vanishes suitably at $k'=0$ and $k'=\infty$.

We cannot use the solution Eq. (III.7) to reduce Eq. (II.8) to a Fredholm problem because by setting $\mu=0$, we have incorrectly treated the low k^2 behavior of Eq. (III.2). However, we should be able to use in Eq. (III.6) the $\mu=0$ resolvent given by Eq. (III.7) to obtain the correct high k^2 behavior of $A_n(k,q)$. This follows from taking the high k^2 limit of Eq. (III.2) and solving.

Setting $I_n(k,q)=B_{0n}(k,q)$ and the carrying out the integrals in Eq. (III.6) we get the following approximation for $A_n(k,q)$

$$A_n(k,q)=\frac{2\pi^2\bar{\lambda}^2(n+1)}{[(n+1)^2+\bar{\lambda}^2]^{1/2}}\left[\frac{1}{\alpha_1}\left(\frac{q}{k}\right)^{\alpha_1}-\frac{1}{\alpha_2}\left(\frac{q}{k}\right)^{\alpha_2}\right],\quad q<k$$
$$=\frac{2\pi^2\bar{\lambda}^2(n+1)}{[(n+1)^2+\bar{\lambda}^2]^{1/2}}\left[\frac{1}{\alpha_1}\left(\frac{k}{q}\right)^{\alpha_1}-\frac{1}{\alpha_2}\left(\frac{k}{q}\right)^{\alpha_2}\right],\quad q>k$$
$$=0,\quad n \text{ odd}. \quad \text{(III.9)}$$

We note that Eq. (III.9) is the solution of Eq. (III.2) with $I=B_{0n}$ and $\mu=0$. However, B_{0n} is also the exact value of the integral (II.10) for B_n with $\mu=0$. Thus the expression Eq. (III.9) for $A_n(k,q)$ is also the exact solution of (II.8) for $\mu=0$. Then combining Eqs. (III.9) and (II.11) we obtain the exact solution to zero mass Eq. (II.1). We could, of course, write down the zero-mass equation directly, but we also wanted to point its connection with high k^2 limit of the finite-mass equation.

IV. THE MASS ZERO SOLUTION

The exact solution of the zero-mass problem is then obtained by inserting the expression for $A_n(k,q)$ given in Eq. (III.9) in Eq. (II.11). After carrying out the summation over n, we must finally continue the result to physical values of k^2, q^2 and $\cos\theta$ to obtain the forward scattering amplitude.

We can write Eq. (II.11) as (calling $\theta_{kq}=\theta$)

$$A(k^2,q^2,\cos\theta)=\sum_{n=2}^\infty A_n(k,q)\left[\frac{C_n(\cos\theta)+C_n(-\cos\theta)}{2}\right] \quad \text{(IV.1)}$$

since $A_n=0$, n odd.

We now use the usual Sommerfield-Watson transformation to evaluate Eq. (IV.1), as done by Regge[3] in the l plane and Bjorken[2] in the n plane.

In order to carry out this transformation we must find a suitable analytic continuation $A(n,k,q)$ of $A_n(k,q)$ to complex n such that

$$A(n,k,q)\sim e^{-\tau\,\mathrm{Re}n}$$

as

$$\mathrm{Re}n\to\infty\quad \tau>0. \quad \text{(IV.2)}$$

This is easily done by choosing the branch cut of the square roots in Eq. (III.9) as follows:

(1) A branch cut from $n=-1-i\bar{\lambda}$ to $n=-1+i\bar{\lambda}$ due to $[(n+1)^2+\bar{\lambda}^2]^{1/2}$.

(2) An additional branch cut from $n=-1+(1-2\bar{\lambda})^{1/2}$ to $n=-1+(1+2\bar{\lambda})^{1/2}$ due to $\alpha_1=[(n+1)^2+1-2((n+1)^2+\bar{\lambda}^2)^{1/2}]^{1/2}$.

(3) An additional branch cut from $n=-1-(1+2\bar{\lambda})^{1/2}$ to $n=-1-(1-2\bar{\lambda})^{1/2}$ due to $\alpha_2=[(n+1)^2+1+2((n+1)^2+\bar{\lambda}^2)^{1/2}]^{1/2}$.

See Fig. 3.

The sign of the square roots are chosen so that α_1 and $\alpha_2\to+\mathrm{Re}n$ as $\mathrm{Re}n\to\infty$. Then if $q>k$, $A(n,k,q)$ satisfies Eq. (IV.2) with $\tau=\ln q/k$.

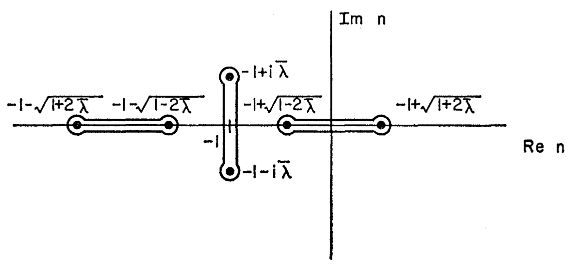

FIG. 3. Cuts in the n plane.

With this definition of $A(n,k,q)$ we can write

$$A(k^2,q^2,\cos\theta)=\frac{1}{4i}\oint_C\frac{dn}{\sin\pi n}A(n,k,q)\times[C_n(\cos\theta)+C_n(-\cos\theta)],\quad\text{(IV.3)}$$

where C is the contour of Fig. 4, which encloses the real n axis greater than two. Using the properties of $C_n(\cos\theta)$ and $A(n,k,q)$ for large $\mathrm{Re}n$ we can deform the contour to a line L parallel to the imaginary axis. The line capital L lies between $n=2$ and $n=-1+(1+2\bar{\lambda})^{1/2}$ which is the position of the singularity of $A(n,k,q)$ which lies farthest to the right in the n plane (see Fig. 5). This deformation is valid provided

$$\mathrm{Im}\theta<\tau=\ln\frac{q}{k}$$

and

$$0<\mathrm{Re}\theta<\pi;$$

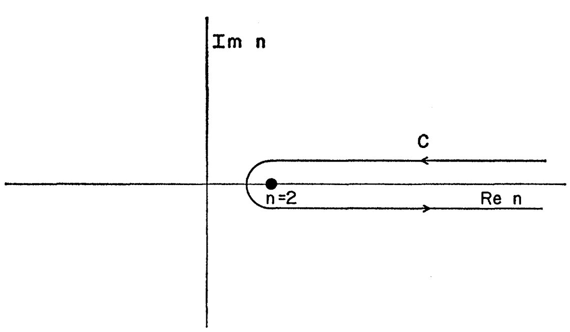

FIG. 4. Contour C.

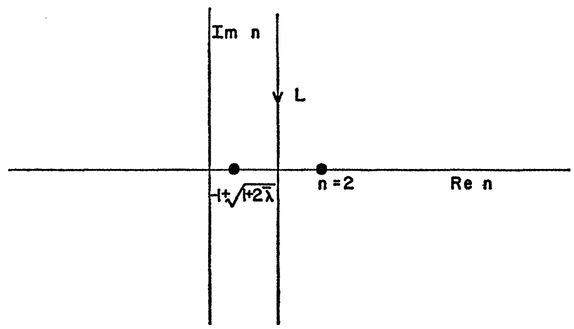

FIG. 5. Contour L.

this is the region in the θ plane for which the integral over the large semicircle can be neglected. After this deformation has been performed, we can analytically continue the resulting integral into the physical region

$$\cos\theta=\frac{E}{\mu}>1,\qquad\text{(IV.4)}$$
$$k^2=q^2=-\mu^2.$$

We let $\cos\theta$ become large by increasing $\mathrm{Im}\theta$ and keeping $\pi>\mathrm{Re}\theta>0$. The integral over L continues to converge and we obtain as our final result for the forward scattering amplitude

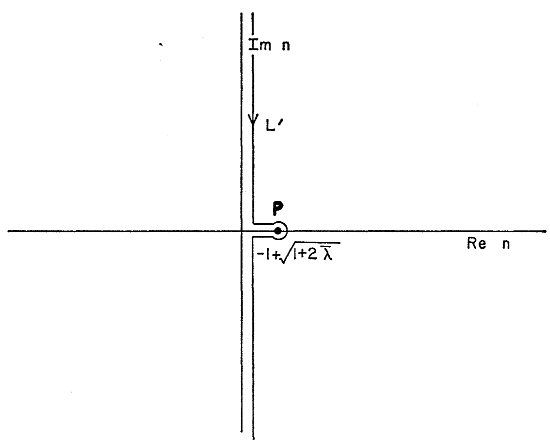

FIG. 6. Contour L deformed.

$$A(E)=\frac{1}{4i}\int_L\frac{dn}{\sin\pi n}A(n)\left[C_n\left(\frac{E}{\mu}\right)+C_n\left(-\frac{E}{\mu}\right)\right],\quad\text{(IV.5)}$$

with

$$A(n)=\frac{2\pi^2\bar{\lambda}^2(n+1)}{[(n+1)^2+\bar{\lambda}^2]^{1/2}}\left[\frac{1}{\alpha_1(n)}-\frac{1}{\alpha_2(n)}\right],\quad\text{(IV.6)}$$

where

$$\alpha_1(n)=\{(n+1)^2+1-2[(n+1)^2+\bar{\lambda}^2]^{1/2}\}^{1/2},$$
$$\alpha_2(n)=\{(n+1)^2+1+2[(n+1)^2+\bar{\lambda}^2]^{1/2}\}^{1/2},$$

with branches as defined above. The single integral (IV.5) over elementary functions is then the sum of bubble-exchange diagrams in the ϕ^4 theory with internal mesons of mass zero. The mass μ of the external mesons must be left nonzero.

V. THE HIGH-ENERGY LIMIT

The contour L can be removed to the left and deformed as it passes the branch point as $n=-1+(1+2\bar{\lambda})^{1/2}$ as in Fig. 6. Now since

$$C_n\left(\frac{E}{\mu}\right)\to\left(\frac{2E}{\mu}\right)^n, \quad E\to\infty\,. \tag{V.1}$$

The leading contribution at high energy to $A(E)$ will come from the portion P of the deformed contour L' which encircles the right end of the branch cut. The discontinuity $\Delta A(n)$ of $A(n)$ across P is from Eq. (IV.6)

$$\Delta A(n)=\frac{4\pi^2\bar{\lambda}^2(n+1)}{[(n+1)^2+\bar{\lambda}^2]^{1/2}}\frac{1}{\{(n+1)^2+1-2[(n+1)^2+\bar{\lambda}^2]^{1/2}\}^{1/2}}\,. \tag{V.2}$$

Let $n+1=(1+2\bar{\lambda})^{1/2}+x$. The contribution of the integral along P to Eq. (IV.5) is

$$A(E)\sim\frac{4\pi^2\bar{\lambda}^2}{4i}\times\int_{-x_0}^{0}\frac{dx[(1+2\bar{\lambda})^{1/2}+x]\{\exp[(-1+(1+2\bar{\lambda})^{1/2}+x)\ln(2E/\mu)]+\exp[(-1+(1+2\bar{\lambda})^{1/2}+x)\ln(-2E/\mu)]\}}{[((1+2\bar{\lambda})^{1/2}+x)^2+\bar{\lambda}^2]^{1/2}[\sin\pi((1+2\bar{\lambda})^{1/2}+x-1)][((1+2\bar{\lambda})^{1/2}+x)^2+1-2[((1+2\bar{\lambda})^{1/2}+x)^2+\bar{\lambda}^2]^{1/2}]^{1/2}}, \tag{V.3}$$

where x_0 is the distance P extends along the branch cut.

As $E\to\infty$, the dominant contribution to (V.3) comes from the neighborhood of $x=0$. We can then approximate Eq. (V.3) in the limit $E\to\infty$ by

$$A(E)\sim\frac{\pi^2\bar{\lambda}^2(1+2\bar{\lambda})^{1/2}}{i(1+\lambda)\sin\pi((1+2\bar{\lambda})^{1/2}-1)}\times\int_{-\infty}^{0}\frac{dx\{\exp[(-1+(1+2\bar{\lambda})^{1/2}+x)\ln(2E/\mu)]+\exp[(-1+(1+2\bar{\lambda})^{1/2}+x)\ln(-2E/\mu)]\}}{\{(1+2\bar{\lambda})^{1/2}[2\lambda x/(1+\bar{\lambda})]\}^{1/2}}, \tag{V.4}$$

$$A(E)\sim\frac{-\pi^{5/2}\bar{\lambda}^{3/2}(1+2\lambda)^{1/4}}{\sqrt{2}(1+\bar{\lambda})^{1/2}}(\sin\pi[(1+2\bar{\lambda})^{1/2}-1])^{-1}\left(\frac{\exp\{[(1+2\lambda)^{1/2}-1]\ln(2E/\mu)\}}{[\ln(2E/\mu)]^{1/2}}+\frac{\exp\{[(1+2\bar{\lambda})^{1/2}-1]\ln(-2E/\mu)\}}{[\ln(-2E/\mu)]^{1/2}}\right). \tag{V.5}$$

Equation (V.5) then gives the high-energy limit of our exact expressions Eq. (IV.5) for $A(E)$. In order to compare with the result of Sawyer we take the weak coupling limit of Eq. (V.5) and get

$$A(E)\sim\frac{-\pi^{3/2}\lambda^{1/2}}{2^{1/2}}\left[\frac{\exp[\lambda\ln(2E/\mu)]}{[\ln(2E/\mu)]^{1/2}}+\frac{\exp[\bar{\lambda}\ln(-2E/\mu)]}{[\ln(-2E/\mu)]^{1/2}}\right], \tag{V.6}$$
$$\bar{\lambda}\to 0,\quad E\to\infty\,.$$

The expression for the high-energy limit Eq. (V.6) in the weak-coupling limit can be obtained from the techniques used by Sawyer when applied to the problem of zero-mass internal pions,[5] and our results in this sense confirm those of Sawyer in the weak-coupling limit. Of course, our results also give the high-energy limit for the problem of zero-mass internal pions for general $\bar{\lambda}$ not in the weak-coupling limit. If our results are compared to Sawyer's results (Ref. 1) in which the mass of the pions is retained in the propagators, the position of the leading singularity in the weak-coupling limit is the same as that of Sawyer; however, the nature of the singularity is different. He obtains a square-root type singularity in the numerator rather than in the denominator of the partial-wave amplitude. Hence, the nonzero, mass

[5] R. Sawyer (private communication).

changes the nature but not the position of the leading singularity.

VI. CONCLUSIONS

(1) We have seen that the exact sum of bubble graphs in the massless ϕ^4 theory has analytic properties in the n plane distinct from the potential theory situation where the Fredholm theory is applicable.[4]

(2) The weak-coupling limit of our result agrees with Sawyer's result for massless pions obtained by summing the most singular terms of perturbation theory. Comparison with Sawyer's result for massive pions shows that neglect of the pion mass does not alter the position of the leading singularity (at least in the weak-coupling limit).

(3) Our third motive which was to reduce a more general problem to a Fredholm equation has been only partially carried out. What remains to be done is to solve Eq. (III.2) with $\mu\neq 0$.

(4) The method of Bjorken seems to be a very useful technique for solving the forward scattering problem. The resulting equations are much simpler than the usual partial-wave equations.

ACKNOWLEDGMENTS

We are deeply indebted to Dr. James Bjorken whose lectures at the University of Washington provided the basis for this work, and to Dr. R. Sawyer for invaluable discussions. One of the authors (M.B.) is grateful to the Sloan Foundation Grant for support.

PHYSICAL REVIEW VOLUME 132, NUMBER 5 1 DECEMBER 1963

Meson Production in $p+d$ Collisions and the $I=0$ π-π Interaction. I. Measured Momentum Spectra of He[3] and H[3] Nuclei from High-Energy $p+d$ Collisions*

ALEXANDER ABASHIAN,† NORMAN E. BOOTH,‡ KENNETH M. CROWE, ROGER E. HILL,‡ AND ERNEST H. ROGERS§
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 17 June 1963)

The momentum spectra of He^3 and H^3 nuclei produced in $p+d$ collisions have been measured with high resolution. Data were obtained at several angles and incident proton energies of about 745 MeV. Each spectrum exhibits a peak due to single pion production and a continuum due to double pion production. An anomalous bump has been observed in the He^3 spectra which we interpret as an $I=0$ virtual di-pion with a scattering length of 2 $\hbar/\mu c$. In this series of four papers we summarize the data obtained over a series of three cyclotron runs. The first paper deals with the experiment and techniques, and the second with the results of the measurements on single-pion production. Next we describe an experiment to determine the spin and parity of the anomalous bump, and finally we discuss its interpretation.

I. INTRODUCTION

It is widely known that in a nuclear or elementary-particle interaction, if one measures the energy spectrum of one type of outgoing particle, information about the nature of the remainder of the outgoing particles can be obtained. In particular, there is a unique (although sometimes double-valued) correspondence between the vector momentum of one of the particles and the total energy in the rest system of the remaining particles. Any stable or resonant value of this total energy is reflected as a bump in the spectrum of the first particle. Referring to the inset of Fig. 1, consider a particle of momentum $\mathbf{p}_1$ striking particle 2 and yielding a particle of momentum $\mathbf{p}_3$ and a group of one or more particles whose momenta add up to $\mathbf{p}_w$. Energy and momentum conservation gives $\mathbf{p}_w=\mathbf{p}_1-\mathbf{p}_3$ and $E_w=E_1+m_2-E_3=(|\mathbf{p}_w|^2+w^2)^{1/2}$, where w is the "mass" or total energy in the rest system defined

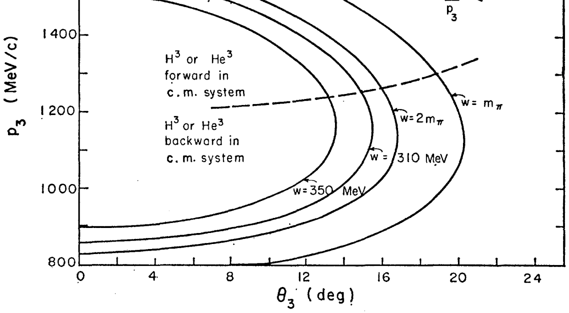

FIG. 1. Kinematical plots of H^3 and He^3 momentum from $p+d$ collisions with 740-MeV protons. The curves are plotted versus the laboratory angle of the H^3 or He^3 for various values of w, the "mass" of the system produced with the H^3 or He^3.

* This work was done under the auspices of the U. S. Atomic Energy Commission.

† Present address: Physics Department, University of Illinois, Urbana, Illinois.

‡ Present address: Enrico Fermi Institute for Nuclear Studies, University of Chicago, Chicago, Illinois.

§ Present address: Aerospace Corporation, El Segundo, California.